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Analytical Frame Work for General Input MWV-VI with Threshold Policy

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Abstract: *Vacation queuing theory is used in computer networking, telecommunication, machine interference. In the present work, analysis of GI/M(n)/1/K multiple working vacation with Vacation interruption has been done. Using the supplementary variable technique and recursive methodthe steady state systems, length distributions at various epochs has been derived. Some performance measure has been evaluated for few parameters. Graphs and table presents the outcome of numerical example considered for this vacation model. Effect of parameter on the performance measure appears from the graphs included in this paper.*

Keywords: *Vacation interruption, Multiple working vacation, supplementary variable, QFSM.*

1. INTRODUCTION

Queueingsystem with vacations makes the queueing system more flexible to use and apply in computer communication system, manufacturing and production systems.There are two types of working vacation, single working vacation and multiple working vacations. Multiple Working vacation was first introduced by Servi and Finn[12].Doshi [4] survey paper gives more details about this theory. Takagi [7]has given extensive literatures on infinite and finite buffer M/G/1 type vacation models. However limited studies have been done on GI/M/1 type models till date. Baba [2] studied GI/M/1 queue with working vacation by the matrix analysis method. Banik et al. [1] worked on GI/M/1/N queue with working vacation and presented the series of numerical results. Li and Tian [8] discussed the discrete time GI/Geo/1 queue with working vacation and vacation interruption. Working vacation policy is practically applicable in optimal design of the system. A details of different type of vacation policy is studied by the book of N. Tian, N and Z. G. Zhang[11]. M. Zhang and Z. Hou [10] discussed the performance analysis of M/G/1 working vacation and vacation interruption.F. Karaesmen, S.M. Gupta[5] investigated the finite capacity GI/M/1 queue with server vacations.U. Chatterjee, S.P. Mukherjee [13] analyzed the model GI/M/1 queue with server vacations.

In this work a model of single – server,multiple working vacation with vacation interruption policy and general arrival process have been analyzed.To economize the cost of the system we allow the server to take multiple vacation in case there is less customer in the system waiting for the service, but it also reduce the working efficiency of the system and cause the loss and dissatisfaction from the customer. Along with the multiple working vacation policy in which customers served with lower service ratethan completely stopping the service. Thus working vacation determines how the server use the vacation period and the vacation interruption The aim of our study is to develop some performance measures such as queue length and waiting time and further using these performance measures we can demonstrate the parameter effect on the measures of the system.

2. DESCRIPTION OF MODEL

Let us Consider a G1/M(n)/1/K/MWV queue with vacation interruption and N policy. We assume that the inter-arrival times of successive arrivals are i.i.d randomvariables with c.d.f. $A(x)$,p.d.f $a(x)$, $x\geq 0$, L.S.T $A^*(\Theta)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$. After completion of busy period server leaves for a working vacation. At a vacation completion epoch, if there are at least N customers waiting in the queue, the server ends his vacation and joins the service at normal service rate; otherwise takes another WV with probability p and vacation interrupt with the probability q. The service times are exponentially distributed during regular busy period and during MWV period with rates μn andηn, $1\leq n \leq K$, respectively, if there are n customers available in the queue before the beginning of a service. The vacation times also follow exponential distribution with rate γn, $1\leq n \leq K$, When there are n customers in the system. Customers are served according to FCFS queue discipline. Let μ, η and γ be the mean service rates during regular busy period, during WV and mean vacation rates respectively. The expressions for μ, η and γ which are mutually independent are given as below:

$$\mu = \sum_{n=1}^{K} \mu_n / K,$$

$$\eta = \sum_{n=1}^{K} \eta_n / K,$$

$$\gamma = \sum_{n=1}^{K} \gamma_n / K,$$

The traffic intensity is given by ρ = λ/μ. The state of the system at time t is described in the following manner:

Ns(t) -the number of customers present in the system including the one who involvesin service,

U(t) -remaining inter-arrival time for the customers just going to enter the system,

$$X(t) - \begin{cases} 0, & \text{if the server is in WV period,} \\ 1, & \text{if the server is in regular busy period.} \end{cases}$$

The joint probabilities are defined as

$J_{n,0}(x, t)dx = P\{ N_s(t) = n, x < U(t) \leq x+dx, X(t) = 0\}$, $x \geq 0$, $0 \leq n \leq K$,

$J_{n,0}(x, t)dx = P\{ N_s(t) = n, x < U(t) \leq x+dx, X(t) = 0\}$, $x \geq 0$, $0 \leq n \leq K$,

As t→∞, the above probabilities are denoted as $J_{n,j}(x)$, j = 0, 1; j ≤ n ≤ K.

3. ANALYSIS OF THE MODEL

In the following section analytic analysis of the model GI/M/1/K/MWV has been done using different analysis methods such as supplementary variable and recursive techniques. The system is stated at two consecutive time epochs t and t+dt and using probabilistic arguments we set up the following differential- difference equations at steady state:

$$-J_{0,0}^{1}(x) = \mu_1 J_{1,1}(x) + \eta_1 J_{1,0}(x),$$
$$-J_{n,0}^{1}(x) = -\eta_n J_{n,0}(x) + a(x) J_{n-1,0}(0) + \eta_{n+1} J_{n+1,0}(x), 1 \leq n \leq N-1,$$
$$-J_{n,0}^{1}(x) = -\delta_n J_{n,0}(x) + a(x) J_{n-1,0}(0) + p\eta_{n+1} J_{n+1,0}(x), N \leq n \leq K-1,$$
$$-J_{K,0}^{1}(x) = -\delta_K J_{K,0}(x) + a(x)\left(J_{K-1,0}(0) + J_{K,0}(0)\right),$$
$$-J_{1,1}^{1}(x) = -\mu_1 J_{1,1}(x) + \mu_2 J_{2,1}(xk),$$
$$-J_{n,1}^{1}(x) = -\mu_n J_{n,1}(x) + a(x) J_{n-1,1}(0) + \mu_{n+1} J_{n+1,1}(x), 2 \leq n \leq N-1$$
$$-J_{n,1}^{1}(x) = -\mu_n J_{n,1}(x) + a(x) J_{n-1,1}(0) + \mu_{n+1} J_{n+1,1}(x) + \gamma_n J_{n,0}(x) + q\eta_{n+1} J_{n+1,0}(x), N \leq n \leq K-1,$$
$$-J_{K,1}^{1}(x) = -\mu_K J_{K,1}(x) + a(x)\left(J_{K-1,1}(0) + J_{K,1}(0)\right) + \gamma_K J_{K,0}(x)$$

Where $\delta_n = \gamma_n + \eta_n (N \leq n \leq K)$ and $J_{n,j}(0)$, j = 0,1; j ≤ n≤ K are respective rates of arrivals.

We assume that Laplace Stieltjes transform of Jn,j(x) be defined as $J^*_{n,j}(\theta) = \int_0^\infty e^{-\theta x} J_{n,j}(x)dx$. Therefore $J_{n,j}(x) \equiv J^*_{n,j}(\theta)$, Given that Jn,j(x) are joint probabilities that there are n customers in the system and the server in state of j at an arbitrary epoch. Finding L.S.T of above set of equations by multiplying all the equations by $e^{-\theta x}$ and integrating with respect to x from 0 to ∞ gives

$$-\theta J_{0,0}^{*}(\theta) = \mu_1 J_{1,1}^{*}(\theta) + \eta_1 J_{1,0}^{*}(\theta) - J_{0,0}(0), \tag{1}$$

$$(\eta_n - \theta) J_{n,0}^{*}(\theta) = \eta_{n+1} J_{n+1,0}^{*}(\theta) + A^{*}(\theta) J_{n-1,0}(0) - J_{n,0}(0), 1 \leq n \leq N-1, \tag{2}$$

$$(\delta_n - \theta) J_{n,0}^{*}(\theta) = A^{*}(\theta) J_{n-1,0}(0) + p\eta_{n+1} J^{*}{}_{n+1,0}(\theta) - J_{n,0}(0), N \leq n \leq K-1, \tag{3}$$

$$(\delta_K - \theta) J_{K,0}^{*}(\theta) = A^{*}(\theta)\left(J_{K-1,0}(0) + J_{K,0}(0)\right) - J_{K,0}(0), \tag{4}$$

$$(\mu_1-\theta)J^*_{1,1}(\theta)=\mu_2 J^*_{2,1}(\theta)-J_{1,1}(0), \tag{5}$$

$$(\mu_n-\theta)J^*_{n,1}(\theta)=\mu_{n+1}J^*_{n+1,1}(\theta)+A^*(x)J_{n-1,1}(0)-J_{n,1}(0), 2\le n\le N-1 \tag{6}$$

$$(\mu_n-\theta)J^*_{n,1}(\theta)=-J_{n,1}(0)+A^*(\theta)J_{n-1,1}(0)+\mu_{n+1}J^*_{n+1,1}(\theta)+\gamma_n J^*_{n,0}(\theta)+q\eta_{n+1}J^*_{n+1,0}(\theta), \tag{7}$$
$N\le n\le K-1,$

$$(\mu_K-\theta)J^*_{n,1}(\theta)=-J_{K,1}(0)+A^*(\theta)\left(J_{K-1,1}(0)+J_{K,1}(0)\right)+\gamma_K J^*_{K,1}(\theta) \tag{8}$$

With the help of above equations, We find the following result.

Lemma 1

$$\sum_{n=0}^{K} J_{n,0}(0)+\sum_{n=1}^{K} J_{n,1}(0)=\lambda \tag{9}$$

The above equation reveals that the mean number of arrivals into the system per unit time is equal to the mean arrival rate λ.

Proof : We obtain the above result by adding the equations (1) to (8) and applying the limit as $\theta\to 0$ and using the normalization condition $\sum_{n=0}^{K} J_{n,0}+\sum_{n=1}^{K} J_{n,1}=1$.

3.1. RELATION BETWEEN STEADY-STATE DISTRIBUTION AT ARBITRARY AND PRE-ARRIVAL EPOCHS

Let $\hat{J}_{n,j}$, j = 0, 1; j ≤ n ≤ K denote the pre- arrival epoch probabilities, i.e., an arrival finds n customers in the system and the server in state j at an arrival epoch. Applying Bayes's theorem, one may have

$$\hat{J}_{n,j}=\frac{J_{n,j}(0)}{\lambda},\ j=0,1\ ;\ j\le n\le K \tag{10}$$

Where λ is given by Lemma 1. To obtain the steady state probabilities at arbitrary epochs, we find relation between pre-arrival epoch and arbitrary epoch probabilities discussed in the following theorem.

Theorem 1The steady state probabilities at arbitrary epochs are given by

$$J_{K,0}=\left(\frac{\lambda}{\delta_K}\right)\hat{J}_{K-1,0} \tag{11}$$

$$J_{n,0}=\frac{\lambda}{\delta_n}\left[\hat{J}_{n-1,0}+\left(\frac{q\eta_{n+1}-\gamma_{n+1}}{\delta_{n+1}}\right)\hat{J}_{n,0}+\sum_{j=n+1}^{K-1}\left(\frac{q\eta_{j+1}-\gamma_{j+1}}{\delta_{j+1}}\right)\times\prod_{s=n+1}^{j}\left(\frac{q\eta_s}{\delta_s}\right)\hat{J}_{s,0}\right],$$
$n = K-1,K-2,...,N, \tag{12}$

$$J_{n,0}=\frac{\lambda}{\eta_n}\left[\hat{J}_{n-1,0}+\left(\frac{\gamma_N}{\delta_N}\right)\hat{J}_{N-1,0}+\sum_{j=N}^{K-1}\left(\frac{q\eta_{j+1}-\gamma_{j+1}}{\delta_{j+1}}\right)\times\prod_{s=N}^{j}\left(\frac{q\eta_s}{\delta_s}\right)\hat{J}_{s,0}+\left(\frac{q\eta_{N+1}-\gamma_{N+1}}{\delta_{N+1}}\right)\hat{J}_{N,0}\right]\quad n=N-1,\dots,1 \tag{13}$$

$$J_{K,1}=\frac{\lambda}{\mu_K}\left[\left(\frac{\gamma_K}{\delta_K}\right)\hat{J}_{K-1,0}+\hat{J}_{K-1,1}\right] \tag{14}$$

$$J_{n,1}=\frac{\lambda}{\mu_n}\left[\hat{J}_{n-1,1}+\left(\frac{\gamma_n}{\delta_n}\right)\hat{J}_{n-1,0}-\sum_{j=n}^{K-1}\left(\frac{q\eta_{j+1}-\gamma_{j+1}}{\delta_{j+1}}\right)\times\prod_{s=N}^{j}\left(\frac{p^{s-n}\eta_s}{\delta_s}\right)\hat{J}_{s,0}\right]$$
n= K-1,K-2,...,N, (15)

$$J_{n,1}=\frac{\lambda}{\mu_n}\left[\hat{J}_{n-1,1}+\left(\frac{\gamma_N}{\delta_N}\right)\hat{J}_{N-1,0}-\sum_{j=N}^{K-1}\left(\frac{q\eta_{j+1}-\gamma_{j+1}}{\delta_{j+1}}\right)\times\prod_{s=N}^{j}\left(\frac{p^{s-n}\eta_s}{\delta_s}\right)\hat{J}_{s,0}\right]$$
n= N-1,N-2,...,2, (16)

$$J_{1,1}=\frac{\lambda}{\mu_1}\left[\left(\frac{\gamma_N}{\delta_N}\right)\hat{J}_{N-1,0}-\sum_{j=N}^{K-1}\left(\frac{q\eta_{j+1}-\gamma_{j+1}}{\delta_{j+1}}\right)\times\prod_{s=N}^{j}\left(\frac{p^{s-n}\eta_s}{\delta_s}\right)\hat{J}_{s,0}\right] \tag{17}$$

$$J_{0,0}=1-\sum_{n=1}^{K}(J_{n,0}+J_{n,1}) \tag{18}$$

Putting $\theta = 0$ in (2) to (8) and using (10), we find the relations (11) to (17) and using the normalization condition $J_{0,0}$ is obtained

3.2 SPECIAL CASES

In this section, we deduced some other models by taking specific values of the parameters p, N, η_n, γ_n and μn.

Case I: If p = N = 1, our model reduces to GI/ M(n)/1/ MWV queue. In this case results match with the results available in Goswami et al.[6].

Case II: For $\eta_n \to 0, \forall n$, the model becomes GI/ M(n)/1/ K queue with multiple vacations and N-policy.Our result match numerically with Chao and Rahman [3].

Case III: If $\mu n = \mu$, $\eta_n \to 0$, $\gamma_n \to \infty$, $\forall$n and q = 1, the model reduces to GI/ M/1/ K queue with N-policy and the results match with those available in Ke and wang [9].

3.3 PERFORMANCE MEASURES

Since the steady state probabilities at various epochs are known, performance measures of the queue can easily be obtained and are given as:

Average number of customers in the system (Ls) = $\sum_{i=1}^{K} i\left(J_{i,0} + J_{i,1}\right)$

Average number of customers in the system when the server is in the service period (Lq) = $\sum_{i=1}^{K}(i-1)\left(J_{i,0} + J_{i,1}\right)$

Average number of customers in the system when the server is on the working vacation (L_{wv}) $= \sum_{i=0}^{K} i\,J_{i,0}$

Average waiting time of a customer in the queue (W_q) = $\frac{Lq}{\lambda^\wedge}$

Average waiting time of a customer in the queue (W_s) = $\frac{Ls}{\lambda^\wedge}$

The probability of loss or blocking$P_{loss} = \hat{J}_{K,0} + \hat{J}_{K,1}$

Where $\lambda^\wedge = \lambda(1 - P_{loss})$ is the effective arrival rate.

3.3.1. WAITING TIME ANALYSIS

In this section we obtain the Laplace- Steiltjes Transform of waiting time distribution of a customer who is served in the system. If W(x) be the waiting time distribution of a customer in the system who is served already then let W*(θ) be its LST then, we have

$$W^*(\theta) = \frac{1}{1-P_{loss}}\left\{\hat{J}_{0,0}\left(\frac{\gamma_n\mu_n}{(\theta+\mu_n)(\theta+\gamma_n+\eta_n)} + \frac{\eta_n}{\theta+\gamma_n+\eta_n} + \sum_{n=1}^{N-1}\hat{J}_{n,1}\left(\frac{\mu_n}{(\theta+\mu_n)}\right)^{n+1}\right)\right.$$
$$\left. + \sum_{n=1}^{N-1}\hat{J}_{n,0}\sum_{K=0}^{n}\left(\frac{\mu_n}{(\theta+\mu_n)}\right)^{n+1-K}\left(\frac{\eta_n}{\theta+\gamma_n+\eta_n}\right)^{K}\left(\frac{\gamma_n}{\theta+\gamma_n+\eta_n}\right) + \sum_{n=1}^{N-1}\hat{J}_{n,0}\left(\frac{\eta_n}{\theta+\gamma_n+\eta_n}\right)^{n+1}\right\}$$

Using this expression one can easily obtain mean waiting time in the system which is given by

$$w = \frac{1}{1-P_{loss}}\left\{\hat{J}_{0,0}\left(\frac{\gamma_n+\mu_n}{(\mu_n)(\gamma_n+\eta_n)}\right) + \sum_{n=1}^{N-1}\hat{J}_{n,1}\left(\frac{n+1}{(\mu_n)}\right)\right.$$
$$\left. + \sum_{n=1}^{N-1}\hat{J}_{n,0}\sum_{K=0}^{n}\left(\frac{k\eta^k\gamma_n}{(\gamma_n+\eta_n)^{k+2}} + \frac{\eta^k\gamma_n}{(\gamma_n+\eta_n)^{k+2}} + \frac{(n+1-k)\eta^k\gamma_n}{(\gamma_n+\eta_n)^{k+1}\mu}\right) + \sum_{n=1}^{N-1}\hat{J}_{n,0}\frac{(n+1)\eta^{n+1}}{(\gamma_n+\eta_n)^{n+2}}\right\}$$

3.3.2 COST MODEL

We develop a total expected cost function per unit time in which the mean service rate during WV(η) is the decision variable. Let

$C_{bp} \equiv$ cost per unit time during regular busy period,

$C_{wv} \equiv$ cost per unit time during WV period,

$C_q \equiv$ cost per unit time for a customer waiting in the queue,

$C_b \equiv$ cost per unit time when a customer is lost due to blocking.

The total expected cost function per unit time according to the definitions of each cost element is given by

Minimize : $C(\eta) = C_{bp}\mu + C_{wv}\eta + C_q L_q + C_b P_{loss}$.

The objective is to determine mean service rate during vacation (η*) to minimize the cost function C(η). We employ QFSM to solve the above optimization problem.

3.4 NUMERICAL RESULTS

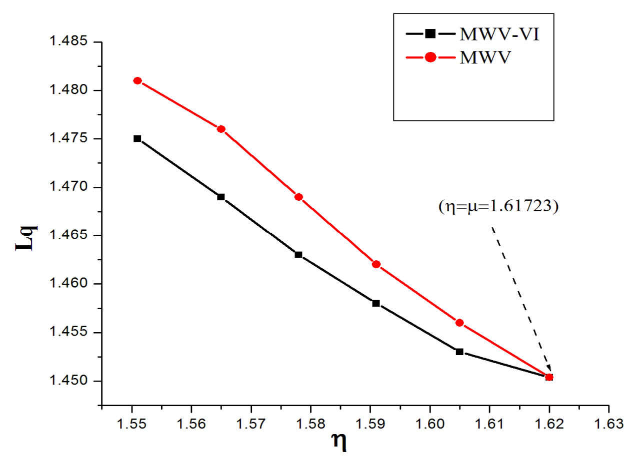

Fig. 1. Impact of η on L_q

To demonstrate the applicability of the results obtained in previous sections, a variety of numerical results have been presented in tables and graphs. To compare the results with Baba [2], we have used the formulae given in [2] to evaluate pre-arrival and arbitrary epoch probabilities of E2/M(n)/1 queue. Using our method we evaluated the results for E2/M(n)/1/12 queue with mean inter-arrival time equal to 0.80862. Since offered load is (ρ = 0.5) < 1 and buffer space is (=12).

TABLE 1: Performance characteristics of E_2/M(n)/1/12 queue with λ=0.80862

	q=0	q=0.2	q=0.4	q=0.6	q=o.8	q=1
L_q	1.22254	1.22261	1.22269	1.22281	1.22292	1.22306
L_s	2.15649	2.15658	2.15668	2.15679	2.15692	2.15708
P_{loss}	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002
W_q	1.51194	1.51204	1.51214	1.51224	1.511236	1.51254
W_s	2.66696	2.66708	2.66721	2.667342	2.66749	2.66765

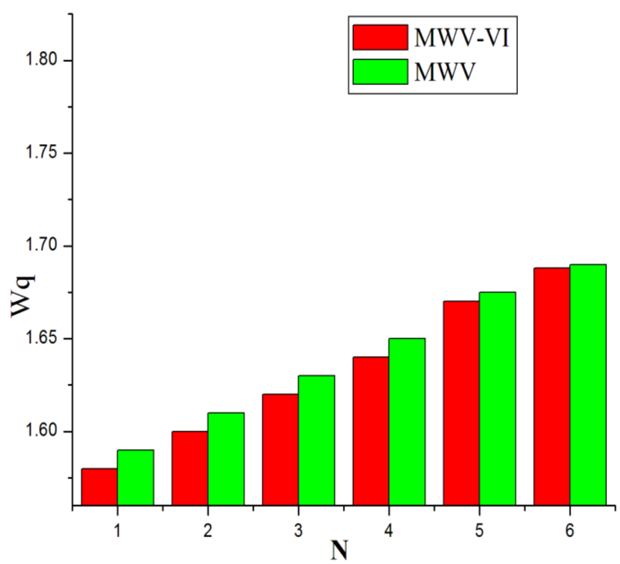

Fig. 2. Impact of Threshold Value (N) on W_q

Table 1 presents the performance characteristics for different q values when inter- arrival times follow Erlang-2 distribution. It can be observed that as q increases, the system characteristics increase and model with vacation interruption performs better than the model without vacation interruption as expected in real world problems. Figure 1 depicts the effect of η on average queue length (Lq) in models with and without vacation interruption for inter-arrival time distribution. We observe that as η increases Lq decreases and Lq converges to the same value as η approaches μ. Further the model with vacation interruption yields lower queue lengths compared to model without vacation interruption.

Figure 2 shows the impact of threshold value (N) on $W_q$when the inter arrival times are exponentially distributed in models. It appears from the figure that W_q increases with the increase of N in both the models with and without vacation interaction.This trend is because as N increases more customers are required for vacation interruption resulting in increase of waiting time.

4. CONCLUSIONS

In this paper, we have carried out an analysis of G1/M (n)/1/K model with multiple working vacation, vacation interruption with N policy that has potential applications in production, manufacturing, traffic signals, telecommunication system and so forth. The steady state system length distribution at various epochs has been evaluated. Along with various performance measures, cost optimization problem is also done using QFSM. Numerical illustrations are reported to demonstrate how the various model parameters of the system influence the behavior of the system. Finally we conclude that queue with vacation interruption and N-policy yield lower queue lengths and waiting time and hence performs better due to the fact of the vacation interruption during working vacation.

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